On Marshall's p-invariant for semianalytic set germs

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ABSTRACT

The invariant $p(V)$ has been introduced by M. Marshall as a measure of the complexity of semialgebraic sets of a real algebraic variety V. This invariant is defined as the least integer such that every semialgebraic set $S \subset V$ has a separating family with $p(V)$ polynomials.

In this paper we provide estimates for the invariant p in the case of analytic set germs. One of the tools we use is a realization theorem which is interesting by itself.

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Introduction

The invariant $p(V)$ has been introduced by M. Marshall as a measure of the complexity of semialgebraic sets of a real algebraic variety V, cf. [Ma1]. Namely, $p(V)$ is the least integer such that every semialgebraic set $S \subset V$ has a separating family with $p(V)$ polynomials. This means that S can be separated from its complement by $p(V)$ polynomials.

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Marshall found upper and lower bounds for $p(V)$ depending on the dimension of V. In this paper we find similar bounds for the invariant p in the case of analytic set germs. One of the tools we use is a realization theorem which is interesting by itself.

The paper is organized as follows: In section 1 we give some definitions and review the results found by M. Marshall in the algebraic case. Section 2 is devoted to spaces of orderings which are a fundamental tool in our work. Finally, in section 3 we state the upper and lower bounds on the p-invariant for analytic set germs and prove a useful realization theorem.

1. The p-invariant for semialgebraic sets

Let $V \subset \mathbb{R}^n$ be an *algebraic set*, i.e., $V = \{x \in \mathbb{R}^n \mid g_1(x) = \ldots = g_r(x) = 0\}$ for some polynomials $g_1, \ldots, g_r \in \mathbb{R}[x_1, \ldots, x_n]$. A *semialgebraic set* S of V is a finite boolean combination of sets of the form $\{x \in V \mid f_1\alpha_1, \ldots, f_m\alpha_m\}$, where $f_1, \ldots, f_m \in \mathbb{R}[x_1, \ldots, x_n]$ and α_i stands for > 0, ≥ 0 or $= 0$. For generalities on semialgebraic sets we refer to [B-C-R] and [Brö1].

We say $\{f_1, \ldots, f_p\} \subset \mathbb{R}[x_1, \ldots, x_n]$ is a *separating family* for a subset $S \subset V$ if $\forall x \in S$ and $\forall y \in V \setminus S$, there is some $i \in \{1, \ldots, p\}$ such that $\operatorname{sign} f_i(x) \neq \operatorname{sign} f_i(y)$. We recall that $\operatorname{sign} b = 1$, 0 or -1 according to the sign of the real number b. If $e = (e_1, \ldots, e_p) \in \{-1, 0, 1\}^p$ and $f = (f_1, \ldots, f_p)$ we define $U(f; e) := \{x \in V \mid \operatorname{sign} f_i(x) = e_i,\ i = 1, \ldots, p\}$. Obviously, $f_1, \ldots, f_p$ is a separating family for S if and only if $S = \cup_{e\in\Delta} U(f; e)$, for some $\Delta \subset \{-1, 0, 1\}^p$. Thus, it is clear that a subset $S \subset V$ has a separating family if and only if S is a semialgebraic set.

The *invariant* $p(V)$ is defined as the least integer such that each semialgebraic subset $S \subset V$ has a separating family of at most $p(V)$ polynomials. Lower and upper bounds in the semialgebraic case are given in the next theorem, cf. [Ma1, Thm.1 and Cor.2].

Theorem 1.1 *Let $V \subset \mathbb{R}^n$ be an algebraic set of dimension d. Then*

$$\log_2(\alpha^{d-1}(2)) + d - 1 \leq p(V) \leq 1 + \sum_{i=1}^{d} (4^{i-1} - 2^{i-1} + 1)$$

where $\alpha(s) = s(s + 1)/2$.

Next table gives these bounds for dimensions $2 \leq d \leq 6$.

$\dim V$	2	3	4	5	6
	$3 \leq p \leq 5$	$5 \leq p \leq 17$	$8 \leq p \leq 75$	$12 \leq p \leq 316$	$20 \leq p \leq 1309$

Table 1

2. Preliminary results on spaces of orderings

It will be useful to study first the p-invariant in the case of spaces of orderings. Then we will come back to the geometric invariant.

Let G be a multiplicative group of exponent 2 with a distinguished element $-1 \neq 1$ and let $\hat{G} = \mathrm{Hom}(G, \mathbb{C})$ be the topological dual group of G when G is endowed with the discrete topology (in fact, in this case, $\hat{G}$ coincides with $\mathrm{Hom}(G, \{-1, 1\})$. Furthermore, let X be a subset of $\hat{G}$. The pair (X, G) is a *prespace of orderings* if the following conditions hold:

O_1: X is closed in $\hat{G}$.

O_2: $\sigma(-1) = -1$ for all $\sigma \in X$.

O_3: $X^\perp := \{g \in G \,|\, \sigma(g) = 1, \forall \sigma \in X\} = \{1\}$.

A *form* ρ *over* G of *dimension* n is an n-tuple $\rho = \langle g_1, \ldots, g_n \rangle$ such that $g_i \in G$. *The signature of* ρ *with respect to* $\sigma \in X$ is defined as $\sigma(\rho) := \sum_{i=1}^n \sigma(g_i) \in \mathbb{Z}$. Two forms ρ and τ are called *similar*, $\rho \sim \tau$, if $\sigma(\rho) = \sigma(\tau)$ for all $\sigma \in X$. If, moreover, $\dim \rho = \dim \tau$ then the two forms are said to be *congruent* and it is written $\rho \equiv \tau$.

The form ρ *represents* $g \in G$ if $\rho \equiv \langle g, g_2, \ldots, g_n \rangle$ for some $g_2, \ldots, g_n \in G$. The set $\{g \in G \,|\, \rho \text{ represents } g\}$ is denoted as $D(\rho)$. Now, a prespace of orderings (X, G) is *a space of orderings* if also the following condition holds:

O_4: Let $\rho = \langle g_1, \ldots, g_n \rangle$, $\tau = \langle h_1, \ldots, h_m \rangle$ be two forms over G and let $a \in D(\rho + \tau) := D(\langle g_1, \ldots, g_n, h_1, \ldots, h_m \rangle)$. Then there exist $g \in D(\rho)$, $h \in D(\tau)$ such that $a \in D(\langle g, h \rangle)$.

Two spaces of orderings (X, G) and (Y, H) are said to be *isomorphic* it there is a group isomorphism $\phi : G \to H$ such that the induced dual isomorphism $\hat{\phi} : \hat{H} \to \hat{G}; \sigma \mapsto \sigma \circ \phi$ maps Y onto X.

For the notion and properties of spaces of orderings we refer to [An-Br-Rz, Ch.IV] and [Bröl]. For a survey on applications of the real spectrum to semialgebraic geometry the reader can take a look at [Be].

Example 2.1 It is possible to associate a space of orderings to any formally real field K. Recall that K is *formally real* if $-1 \notin \Sigma := \{a \in K \,| a = a_1^2 + \ldots + a_r^2$, for some $r \in \mathbb{N}$ and $a_1, \ldots, a_r \in K\}$ or, in other words, if the field K can be ordered. Let X be the *real spectrum of* K (usually written as $\mathrm{Spec}_r K$), that is, the set of orderings of K compatible with the field structure in the usual way and let G be the quotient group $G := K^*/\Sigma^*$. For $\sigma \in X$ and $g \in K^*$ we define $\sigma(g) = +1$ (resp., -1) if $g > 0$ (resp., $g < 0$) in the ordering σ. Obviously, $\sigma(g) = \sigma(gt)$ if $t \in \Sigma^*$ so it makes sense to define $\sigma(g\Sigma^*)$ as $\sigma(g)$. Then σ can be seen as an element of $\hat{G}$ and it can be checked that (X, G) has a structure of space of orderings, see [An-Br-Rz, Ex.IV.1.4].

In the case $K = \mathbb{R}$ we have that $X = \{\sigma\}$, where σ is the unique ordering of $\mathbb{R}$ and $G \simeq \mathbb{Z}_2$. This space is called atomic space and denoted by E. □

A *basic set* of a space of orderings (X, G) is a subset of X of the form $\{\sigma \in X \mid \sigma(g_1) > 0, \ldots, \sigma(g_r) > 0\}$, for some $g_1, \ldots, g_r \in G$. The least integer $s \geq \infty$ such that any basic set of X can be described by s elements of G is called the *stability index* of (X, G). The *constructible sets* of X are the finite boolean combinations of basic sets.

A *separating family for* $S \subset X$ is a subset $\{g_1, \ldots, g_r\} \subset G$ such that if $\sigma \in S$ and $\sigma' \in X \setminus S$ then there exists some $i \in \{1, \ldots, r\}$ such that $\sigma(g_i) \neq \sigma'(g_i)$. As in the algebraic case a subset $S \subset X$ has a separating family if and only if S is a constructible set. The invariant $p(X, G)$ is defined as the least integer such that every constructible set of X has a separating family of at most $p(X, G)$ elements of G.

Now, we can state the following fundamental result, cf. [Ma1, Cor.1]. It is worth to reproduce the proof here.

Theorem 2.2 *Let (X, G) be a space of orderings with stability index s. Then $p(X, G) \leq 4^{s-1} - 2^{s-1} + 1$.*

Proof: Given a constructible subset $S \subset X$ there exists a form ϕ of dimension 4^{s-1} such that $\sigma(\phi) = \left\{ \begin{array}{lll} 2^{s-1} & \text{if} & \sigma \in S \\ -2^{s-1} & \text{if} & \sigma \in X \setminus S \end{array} \right.$, cf. [Bröl, Prop.5.24].

Let $\phi = \langle a_1, \ldots, a_{4^{s-1}} \rangle$ with $a_i \in G$. Then the number of the a_i's which are positive in σ is $(4^{s-1} + 2^{s-1})/2$ when $\sigma \in S$ and $(4^{s-1} - 2^{s-1})/2$ when $\sigma \in X \setminus S$. Therefore there is a separating family for S with $q_s = 4^{s-1} - 2^{s-1} + 1$ elements and we can suppose that the first q_s elements of ϕ, i.e., $a_1, \ldots, a_{q_s}$, form such a separating family. Moreover, we can write $S = \cup_{e \in \Delta_s} U(a; e)$ with $\Delta_s = \{e \in \{-1, 1\}^{q_s} \mid e \text{ has at least } 1 + (4^{s-1} - 2^{s-1})/2 \text{ components equal to } 1\}$, $a = (a_1, \ldots, a_{q_s})$ and $U(a; e) := \{\sigma \in X \mid \sigma(a_i) = e_i,\ i = 1, \ldots, q_s\}$. □

Remark 2.3 Note that in the description of S as $\cup_{e \in \Delta_s} U(a; e)$, the subset $\Delta_s \subset \{-1, 1\}^{q_s}$ is the same for any space of orderings with stability index s and it does not depend on S. This fact will be important later. Of course, what characterizes S are the elements a_i.

3. The p-invariant for semianalytic set germs

In this section we will be interested in analytic set germs in analytic manifolds, which can always reduce to the case of the affine space $\mathbb{R}^n$. The ring of germs of analytic funtions at $0 \in \mathbb{R}^n$ will be denoted as $\mathcal{O}_n$. The set germ of $\{f_1 = \ldots = f_r = 0\}$ for some $f_1, \ldots, f_r \in \mathcal{O}_n$ is called an *analytic set germ*. If X_0 is such an analytic set germ its ideal is defined as $\mathcal{J}(X_0) := \{f \in \mathcal{O}_n \mid f = 0 \text{ on } X_0\}$. The reduced ring $\mathcal{O}(X_0) := \mathcal{O}_n / \mathcal{J}(X_0)$ is the ring of *analytic function germs of* X_0. A *semianalytic set germ* of X_0 is a finite boolean combination of set germs of the form $\{f_1\alpha_1, \ldots, f_m\alpha_m\}$, where $f_1, \ldots, f_m \in \mathcal{O}(X_0)$ and α_i stands for > 0, ≥ 0 or $= 0$. We refer to [An-Br-Rz, VIII.2] for background on analytic and semianalytic set germs.

An analytic set germ X_0 is called *irreducible* if it is not the union of two analytic set germs strictly contained in X_0. In that case $\mathcal{I}(X_0)$ is a prime ideal of $\mathcal{O}_n$. Every analytic germ X_0 has an irredundant decomposition $X_0 = X_0^{(1)} \cup \ldots \cup X_0^{(r)}$, where all the $X_0^{(i)}$ are irreducible analytic set gems. The *Zariski closure* of a subset $S \subset X_0$ is the minimal analytic germ $\overline{S}^z$ such that $S \subset \overline{S}^z$ and we put $\dim S = \dim \overline{S}^z$. Two subsets $S, S' \subset X_0$ are said to be *generically equal*, which we write as $S \overset{g}{=} S'$, if $\dim X_0 \cap ((S \cup S') \setminus (S \cap S')) < \dim X_0$, that is, if they differ in something of codimension strictly greater than 0.

3.1. Upper bounds

The p-invariant of an analytic set germ X_0 is defined as in the algebraic case, cf. section 1, with the only difference that the separating functions are elements of $\mathcal{O}(X_0)$. For the upper bounds we have the following result which is slightly better than the corresponding one in the algebraic case.

Theorem 3.1 *Let X_0 be an analytic set germ of dimension d. Then $p(X_0) \leq \sum_{i=1}^{d}(4^{i-1} - 2^{i-1} + 1)$.*

Proof: By induction on $\dim X_0$.

If $\dim X_0 = 1$ then any semianalytic subset $S \subset X_0$ is a finite union of half-branches and possibly the point $\{0\}$. Thus one function suffices to separate S from its complement, cf. [An-DC, Lem.1.2] or [Rz]. We recall that an irreducible analytic curve germ is the disjoint union of two half-branches and the point $\{0\}$.

Suppose that X_0 is an irreducible analytic set germ of dimension $d \geq 2$ and let $S = \cup_i \{f_{i1}\alpha_{i1}, \ldots, f_{ir_i}\alpha_{ir_i}\}$ (α_{ij} stands for > 0, < 0 or $= 0$) be a semianalytic subset germ of X_0. Let $\tilde{S}$ be the constructible set of the real spectrum of the field of fractions of $\mathcal{O}(X_0)$, which will be denoted as $\mathcal{K}(X_0)$, defined by the same formula, i.e., $\tilde{S} := \cup_i \{\sigma \in \operatorname{Spec}_r \mathcal{K}(X_0) \,|\, \sigma(f_{i1})\alpha_{i1}, \ldots, \sigma(f_{ir_i})\alpha_{ir_i}\}$.

The stability index of $\operatorname{Spec}_r \mathcal{K}(X_0)$ is equal to d, cf. [An-Br-Rz, Ch.VII and VIII], so $\tilde{S}$ has a separating family with $q_d = 4^{d-1} - 2^{d-1} + 1$ elements, cf. Theorem 2.2. Then we have $\tilde{S} = \cup_{e \in \Delta_d} \tilde{U}(f_1, \ldots, f_{q_d}; e)$, with $\tilde{U}(f_{i1}, \ldots, f_{iq_d}; e) = \{\sigma \in \operatorname{Spec}_r \mathcal{K}(X_0) \,|\, \sigma(f_i) = e_i,\ i = 1, \ldots, q_d\}$ and Δ_d as in theorem 2.2.

Let $T = \cup_{e \in \Delta_d} U(f_1, \ldots, f_{q_d}; e)$ with $U(f_1, \ldots, f_{q_d}; e) = \{x \in X_0 \,|\, \operatorname{sign} f_i(x) = e_i,\ i = 1, \ldots, q_d\}$. Clearly $\tilde{S} = \tilde{T}$ so $S \overset{g}{=} T$, cf. [Bröl, Prop.3.4]. Hence, the analytic germ $X_0' := \overline{(S \setminus T) \cup (T \setminus S)}^z \subset X_0$ has dimension stricly smaller than d. By the induction hypothesis $S \cap X_0'$ has a separating family $\{h_1, \ldots, h_m\}$ with $m \leq \sum_{i=1}^{d-1}(4^{i-1} - 2^{i-1} + 1)$.

Let b be a *positive equation of* X_0', that is, $X_0' = \{x \in X_0 \,|\, b(x) = 0\}$ and $b > 0$ on $X_0 \setminus X_0'$. It can be checked that $\{bf_1, \ldots, bf_{q_d}, h_1, \ldots, h_m\}$ is a separating family for S whence the result follows.

Suppose now that X_0 is an analytic germ of dimension $d \geq 2$, possibly reducible, and let $X_0^{(1)}, \ldots, X_0^{(r)}$ be its irreducible components of dimension d. If $S \subset X_0$ is a semianalytic set germ then, as we have seen above, there are $f_{ij} \in \mathcal{O}(X_0)$, $i = 1, \ldots, r$, $j = 1, \ldots, q_d$, such that:

$$S \cap X_0^{(i)} \overset{g}{=} \cup_{e \in \Delta_d} U(f_{i1}, \ldots, f_{iq_d}; e) \cap X_0^{(i)}.$$

Moreover, the f_{ij} can be chosen so that $f_{ij} \notin \mathcal{J}(X_0^{(i)})$.

Define $f_j := \sum_{i=1}^{r} g_i^2 f_{ij}$, $j = 1, \ldots, q_d$, where $g_i \in \cap_{k \neq i} \mathcal{J}(X_0^{(k)}) \setminus \mathcal{J}(X_0^{(i)})$, $i = 1, \ldots, r$, and $S' := \cup_{e \in \Delta_d} U(f_1, \ldots, f_{q_d}; e)$. Then it can be checked that $S \cap X_0^{(i)} \overset{g}{=} S' \cap X_0^{(i)}$, $i = 1, \ldots, r$ and so the germ $X_0' := \overline{(S \setminus S') \cup (S' \setminus S)}^{z}$ has dimension strictly smaller than d. Thus, by the induction hypothesis, there exists a separating family $\{h_1, \ldots, h_m\}$ with $m \leq \sum_{i=1}^{d-1}(4^{i-1} - 2^{i-1} + 1)$ for $S \cap X_0'$ in X_0'. Let b be a positive equation of X_0'. Then $\{bf_1, \ldots, bf_{q_d}, h_1, \ldots, h_m\}$ is a separating family for S in X_0 and we are done. □

Example 3.2 Theorem 3.1 states that $p(X_0) = 1$ if $\dim X_0 = 1$. To see the difference with the algebaic case consider, for example, $V = \mathbb{R}$ and $S = \{-1 \leq x \leq 0\} \cup \{1 < x < 2\}$. Suppose that one polynomial f separates S from $\mathbb{R} \setminus S$. Then f must change sign at 0 since it separates points at both sides of 0. Thus f vanishes at 0. By a similar argument f vanishes at 1. But $0 \in S$ and $1 \notin S$ so f cannot separate these two points. Hence a separating family for S needs at least two polynomials. In fact, $p(V) = 2$ if V is a real algebraic variety of dimension 1. Of course, such an example cannot exist in the case of a one-dimensional analytic set germ.

3.2. Finite spaces of orderings

Before we can state the lower bounds we will collect some facts about finite spaces of orderings to be used later.

Let (X, G) be a space of orderings and let $Y \subset X$, $H \subset G$ such that $H = Y^{\perp} := \{g \in G \,|\, \sigma(g) = 1, \forall \sigma \in Y\}$ and $Y = H^{\perp} := \{\sigma \in X \,|\, \sigma(h) = 1, \forall h \in H\}$. Then $(Y, G/H)$ is again a space of orderings which is called a *subspace of* (X, G), cf. [Ma2, Thm.2.2]. Sometimes we omit any reference to H and just say that Y is a subspace of X. In that case it is understood that $H = Y^{\perp}$ and that the condition $Y = H^{\perp} = Y^{\perp\perp}$ is satisfied.

There are two constructions with spaces of orderings which give new spaces of orderings: addition and extension.

If (X_1, G_1) and (X_2, G_2) are spaces of orderings define $(X, G) = (X_1 \cup X_2, G_1 \times G_2)$ with the distinguished element $(-1, -1) \in G$ and the action $\sigma_1(g_1, g_2) = \sigma_1(g_1)$, $\sigma_2(g_1, g_2) = \sigma_2(g_2)$, where $\sigma_i \in X_i$, $i = 1, 2$ and $(g_1, g_2) \in G$. Then (X, G) is again a space of orderings which is called *the sum of* (X_1, G_1) and (X_2, G_2) and denoted by $(X_1, G_1) + (X_2, G_2)$. For brevity, sometimes we will denote the sum simply as $X_1 + X_2$. It is clear that, for example, (X_1, G_1) can be considered as a subspace of $(X, G) = (X_1, G_1) + (X_2, G_2)$ taking $Y = X_1$, $H = \{1\} \times G_2$ and identifying G_1 with $G_1 \times G_2/H$.

Let (X', G') be a space of orderings, H a group of exponent 2 endowed with the discrete topology and let $(X, G) = (\hat{H} \times X', H \times G')$ with distinguished element $(1, -1) \in G$ and the action $(\alpha, \sigma)(h, g) = \alpha(h)\sigma(g)$, $(\alpha, \sigma) \in (\hat{H} \times X')$ and $(h, g) \in G$. Then (X, G) is a space of orderings which is called *the extension of* (X', G') *by* H and denoted as $(X', G')[H]$ or simply $X'[H]$, cf. [An-Br-Rz, IV.2.13]. The space of orderings (X', G') can be naturally embedded as a subspace of $(X', G')[H]$. Just note that $(\{1\} \times X')^\perp = H \times \{1\}$ and $\{1\} \times X' = (H \times \{1\})^\perp$.

The stability index of sums and extensions behaves in a simple way. Namely, $s(X_1 + X_2) = \max\{s(X_1), s(X_2)\}$ unless $X_1 = X_2 = E$ (the atomic space defined in example 2.1). In that case $s(E) = 0$ but $s(E + E) = 1$, cf. [An-Br-Rz, IV.2.2]. For extensions, we have that $s(X'[H]) = s(X') + \dim_{\mathbb{F}_2}(H)$, cf. [An-Br-Rz, IV.2.14].

A space of orderings (X, G) is called a *finite space of orderings*, if it is built up, starting from atomic spaces, by finitely many additions and extensions. The construction of a space of finite type is essentially unique. The only ambiguity lies in the isomorphism $E + E \simeq E[\mathbb{Z}_2]$ and the derived ones, as for example $(E + E)[H] \simeq E[\mathbb{Z}_2 \times H]$, cf. [An-Br-Rz, Ch.IV].

Example 3.3 Let X_0 be the set germ of $\mathbb{R}^2$ at the origin which we simply write $X_0 = \mathbb{R}^2$ and let us call K to the field of fractions of $\mathcal{O}(X_0)$, that is, the field of meromorphic functions germs at 0. The real spectrum $X = \operatorname{Spec}_r K$ is a space of orderings with group $G = K^*/\Sigma^*$, cf. example 2.1. Let us take the half-branch α_1 (see figure 4 below), which is the germ of the subset $\{x = 0\} \cap \{y > 0\}$. According to the both sides of α_1 there are two generizations α_{11} and α_{12} which are orderings of K. For example, a meromorphic function germ is positive in α_{11} if it is positive to the right of α_1.

Let us call P_{11} to the cone of positive meromorphic function germs and the function germ 0. Then taking $Y_1 = \{\alpha_{11}\}$ and $H_1 = P_{11}^*/\Sigma^*$ it is easy to check that $(Y_1, G/H_1)$ is an atomic space which turns out to be a subspace of (X, G).

In a similar way, consider the half-branch α_2, see again figure 4, and one of its generizations, say, α_{21}. As before $(Y_2, G/H_2)$ with $Y_2 = \{\alpha_{21}\}$ and $H_2 = P_{21}^*/\Sigma^*$ is an atomic space and a subspace of (X, G). It can also be checked that $(Y, G/H)$ is a subspace of (X, G) with $Y = \{\alpha_{11}, \alpha_{21}\}$ and $H = P^*/\Sigma^*$, where $P^* = \{g \in K^* \mid \alpha_{11}(g) = \alpha_{21}(g) = +1\}$. It is clear that the quotient group $G/H \simeq K^*/P^*$ has four classes, namely, $\{g \in K^* \mid \alpha_{11}(g) = e_1, \alpha_{21}(g) = e_2\}$ with $(e_1, e_2) \in \{-1, +1\}^2$ so $G/H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus, $(Y, G/H)$ is isomorphic to the sum $(Y_1, G/H_1) + (Y_2, G/H_2)$. Of course, as said before, it is also isomorphic to $E[\mathbb{Z}_2]$.

Consider now $Y' = \{\alpha_{11}, \alpha_{12}, \alpha_{31}, \alpha_{32}\}$, that is, the four generizations of two half-branches, α_1 and α_3, of the same irreducible curve germ. Then it can be seen that $Y' \simeq E[\mathbb{Z}_2^2]$ or what is the same $Y' \simeq (E + E)[\mathbb{Z}_2]$.

As a last example consider $Y'' = \{\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}\}$, that is, the four generizations of two half-branches, α_1 and α_2, which are independent (i.e., belonging to different irreducible curve germs). Then it can be seen that $Y'' \simeq (E+E)+(E+E)$ or $Y'' \simeq E[\mathbb{Z}_2] + E[\mathbb{Z}_2]$.

Concerning stability indexes, using the above formulas we have $s(Y_1) = 0$, $s(Y_2) = 0$, $s(Y) = 1$, $s(Y') = 2$ and $s(Y'') = 1$. In particular, the finite spaces of orderings Y' and Y'' are not isomorphic. $\square$

The folllowing result involving finite spaces of orderings will be used below to obtain lower bounds, cf. [Ma1, Thm.6]. For $s \geq 2$ we define p_s as the least integer $\geq \log_2(\alpha^{s-1}(2)) + s - 1$, where $\alpha(n) = n(n+1)/2$. We also define $p_1 := 1$.

Theorem 3.4 *Let $s \geq 1$ an integer. There exists a finite space of orderings (X, G) with stability index s and a subset $S \subset X$ such that any separating family for S requires p_s elements, i.e., $p(X, G) \geq p_s$.*

3.3. Lower bounds

We will state the following *realization theorem* which will allow us to get the lower bounds by applying Theorem 3.4.

Theorem 3.5 (Realization Theorem) *Let $X_0 \subset \mathbb{R}^n$ be an irreducible analytic set germ of dimension $d \geq 2$. Then any finite space of orderings with stability index $\leq d$ can be realized as a subspace of Spec$_r \mathcal{K}(X_0)$.*

Proof: By induction on d.

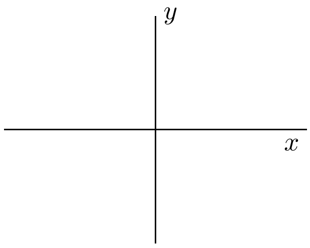
Suppose first $d = 2$ and $X_0 = \mathbb{R}^2$. If (X, G) is a finite space of orderings of stability index equal to 2 then it can be decomposed, cf. [An-Br-Rz, Ch.IV], as a sum $(X, G) = (X_1, G_1) + \cdots + (X_r, G_r)$, where each (X_i, G_i) is an atomic space or an extension with stability index at most 2.

If (X_i, G_i) is an atomic space then we take a generization of a half-branch of X_0, see example 3.3, in such a way that different atomic spaces correspond to independent half-branches.

If (X_j, G_j) is an extension of $(\overline{X}_j, \overline{G}_j)$ with $|G/\overline{G}_j| = 2$, then $(\overline{X}_j, \overline{G}_j)$ must be a sum of atomic spaces, say, $(Y_1, H_1) + \cdots + (Y_n, H_n)$. Now, we blow up the origin along a line l_j and take half-branches $\alpha_1, \ldots, \alpha_n$ at different points $P_1, \ldots, P_n$ on the exceptional divisor (see figures 1 and 2, where a blow up along the y-axis is representated, being the line $\{x' = 0\}$ the exceptional divisor). For the space (X_j, G_j) we take the set of generizations $\{\alpha_{11}, \alpha_{12}, \ldots, \alpha_{n1}, \alpha_{n2}\}$.

To see that, for example, α_{11} can be considered as an ordering of $\mathcal{K}(X_0)$ take any $f(x, y) \in \mathcal{O}(X_0)$. Then $g(x', y') := f(x', x'y')$ can be seen as an analytic function germ at P_1 and so it has a definite sign in α_{11}. Thus it makes sense to talk of the sign of f in α_{11}. For a meromorphic function germ f/g we just recall that sign $(f/g) =$ sign (fg).

Now, it can be checked that the set of all those orderings is a realization of the finite space of orderings (X, G) as a subspace of $\operatorname{Spec}_r \mathcal{K}(X_0)$.



$x = x'$
$y = x'y'$

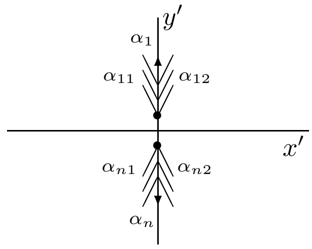


Figure 1 *Figure 2*

If $d = 2$ and X_0 is not regular, we will take its desingularization. More precisely, let $X \subset M$ be an analytic set inducing the germ X_0 and M an analytic manifold. Then there is an analytic mapping, cf. [Bi-Mi], $\pi : M' \to M$ such that $\pi^{-1}(X) = X' \cup \pi^{-1}(\operatorname{Sing} X)$, where X' is a smooth analytic subspace of M', the *strict transform* of X. Moreover, $\pi : X' \to X$ restricts to a proper mapping of X' onto X such that $\pi : X' \setminus \pi^{-1}(\operatorname{Sing} X) \to X \setminus \operatorname{Sing} X$ is an isomorphism.

Now, if $0' \in X'$ is a preimage of 0, that is, $\pi(0') = 0$ then π induces a mapping between the set germs $\pi : X'_{0'} \to X_0$ and we have the associated homomorphism between the fields of fractions $\pi^* : \mathcal{K}(X_0) \to \mathcal{K}(X'_{0'}); f \mapsto f \circ \pi$. Finally, there is an induced mapping, which we call again π, between the real spectra $\pi : \operatorname{Spec}_r \mathcal{K}(X'_{0'}) \to \operatorname{Spec}_r \mathcal{K}(X_0); \sigma \mapsto \sigma \circ \pi^*$. This mapping π is injective as can be deduced from the fact that orderings and ultrafilters of open set germs can be identified in this case, cf. [An-Br-Rz, VIII.2.5]. Since $X'_{0'}$ is regular, the results follows from what we have seen above.

If $d \geq 3$ and (T, G) is a finite space of orderings of stability index d then $(T, G) = (T_1, G_1) + \cdots + (T_r, G_r)$, where (T_i, G_i) is an atomic space or an extension of $(\overline{T}_i, \overline{G}_i)$ with $|G/\overline{G}_i| = 2$ and $s(\overline{T}_i, \overline{G}_i) = d - 1$, cf. [An-Br-Rz, Ch.IV]. If (T_i, G_i) is an atomic space we define $(\overline{T}_i, \overline{G}_i) = (T_i, G_i)$.

Let Y_i be a divisor of X_0, cf. [An-Br-Rz, V.4.1]. By the induction hypothesis $(\overline{T}_i, \overline{G}_i)$ can be realized as a subspace of $\operatorname{Spec}_r \mathcal{K}(Y_i)$. Consider now the two generizations of each ordering of $(\overline{T}_i, \overline{G}_i)$ if (T_i, G_i) is an extension or only one generization in case (T_i, G_i) is an atomic space. Then the collection of these generizations, for $i = 1, \ldots, r$, is a realization of (T, G) as a subspace of $\operatorname{Spec}_r \mathcal{K}(X_0)$. $\square$

We point out that similar results have been achieved in a semialgebraic context. To the best of our knowledge, the first realization theorem is due to Bröcker, cf. [Brö2, Prop.3.3]. Marshall refined it in [Ma1, Thm.7].

Remark 3.6 The realization theorem cannot be extended to dimension 1. If $\dim X_0 = 1$ then X_0 is a union of n irreducible curve set germs and $\operatorname{Spec}_r \mathcal{K}(X_0)$ is a sum of $2n$ atomic spaces. Thus, the spaces of orderings which can be realized as a subspace of $\operatorname{Spec}_r \mathcal{K}(X_0)$ are sums of at most $2n$ atomic spaces. □

As a consequence of the realizaton theorem and theorem 3.4 we have the following lower bounds.

Proposition 3.7 *Let X_0 be an analytic set germ of dimension $d \geq 2$ and let $\alpha(n) = n(n+1)/2$. Then*

$$p(X_0) \geq \log_2(\alpha^{d-1}(2)) + d - 1.$$

Proof: By theorem 3.4 there is a finite space of orderings (X', G') with stability index d such that $p(X', G') = p_d \geq \log_2(\alpha^{d-1}(2)) + d - 1$. If X_0 can be decomposed as $X_0 = X_0^{(1)} \cup \ldots \cup X_0^{(r)}$ where each $X_0^{(i)}$ is irreducible and, say, $\dim X_0^{(1)} = d$. By the previous theorem (X', G') can be realized as a subspace of $\operatorname{Spec}_r \mathcal{K}(X_0^{(1)})$.

Consider a subset S of (X', G') such that any separating family for S requires p_d elements and take a semianalytic set germ $T \subset X_0^{(1)}$ such that $S \subset \tilde{T}$ and $(X' \setminus S) \cap \tilde{T} = \emptyset$, where $\tilde{T}$ is the constructible set of $\operatorname{Spec}_r \mathcal{K}(X_0^{(1)})$ defined by the same formula than T. Then a separating family for T requires p_d elements. □

These lower bounds say, in particular, that $p(\mathbb{R}^2) \geq 3$. Next example gives a direct proof of this fact.

Example 3.8 *Let $S \subset X_0$ be any semianalytic germ such that $\alpha_1, \alpha_2 \in S$, α_{11}, α_{12}, α_{21}, $\alpha_{31} \in \tilde{S}$ and $\alpha_3, \alpha_4 \notin S$, $\alpha_{22}, \alpha_{32}, \alpha_{41}, \alpha_{42} \notin \tilde{S}$, see figure 3. For example, we can take $S = \{x + y \geq 0, y \geq 0\} \cup \{x > 0, x + y \leq 0\}$. Then any separating family for S has at least 3 elements.*

For suppose there is a separating family with 2 elements f_1, $f_2 \in \mathcal{O}(X_0)$. One of these functions, say f_1 will separate α_{31} and α_{32}. Thus, f_1 will vanish on the half-branches α_1 and α_3, changing sign at both half-branches. Hence, f_2 must separate α_1 and α_3 and, in particular, it will not vanish along them. Altogether we can suppose that: $\alpha_{11}(f_1) = +1$, $\alpha_{12}(f_1) = -1$, $\alpha_{31}(f_1) = +1$, $\alpha_{32}(f_1) = -1$ and $\alpha_{11}(f_2) = +1$, $\alpha_{12}(f_2) = +1$, $\alpha_{31}(f_2) = -1$, $\alpha_{32}(f_1) = -1$.

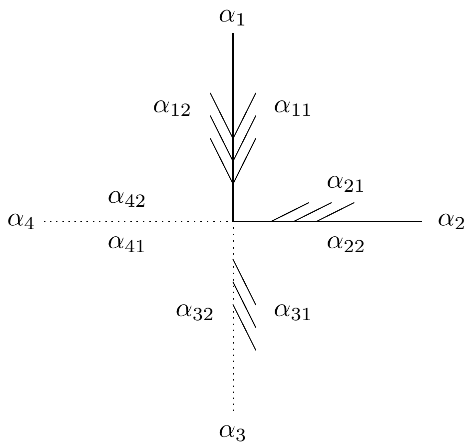


Figure 3

Suppose that f_1 separates α_{21} and α_{22}. Then it would also separate α_{41} and α_{42} so it would be positive in one of these orderings, say in α_{41}. Thus $\alpha_{11}(f_1) = \alpha_{41}(f_1)$ and $\alpha_{31}(f_1) = \alpha_{41}(f_1)$. Now, if $\alpha_{41}(f_2) = +1$ then $\alpha_{11}(f_2) = \alpha_{41}(f_2)$ whereas if $\alpha_{41}(f_2) = -1$ we have $\alpha_{31}(f_2) = \alpha_{41}(f_2)$. Therefore $\{f_1, f_2\}$ cannot be a separating family for S.

Finally, if f_2 is the function which separates α_{21} and α_{22}, then it would separate α_{41} and α_{42} and it would be positive in one of these orderings, say in α_{41}. Thus $\alpha_{11}(f_2) = \alpha_{41}(f_2)$ and $\alpha_{12}(f_2) = \alpha_{41}(f_2)$. Now, if $\alpha_{41}(f_1) = +1$ then $\alpha_{11}(f_1) = \alpha_{41}(f_1)$ whereas if $\alpha_{41}(f_1) = -1$ we have $\alpha_{12}(f_1) = \alpha_{41}(f_1)$. Therefore $\{f_1, f_2\}$ cannot be a separating family for S. □

References

[An-Br-Rz] C. Andradas, L. Bröcker, J. Ruiz: *Constructible sets in real geometry.* Ergeb. Math. **33,** Springer-Verlag, Berlin 1996.

[An-DC] C. Andradas, A. Díaz–Cano: *Closed stability index of excellent henselian local rings.* To appear in Math. Z..

[Be] E. Becker: *On the real spectrum of a ring and its application to semialgebraic geometry.* Bulletin AMS **15** (1986), 19–60.

[Bi-Mi] E. Bierstone, P.D. Milman: *Local resolution of singularities.* Lecture Notes in Math. **1420** (1990), 42–64.

[B-C-R] J. Bochnak, M. Coste, M.F. Roy: *Real algebraic geometry.* Ergeb. Math. **36,** Springer-Verlag, Berlin 1998.

[Brö1] L. Bröcker: *On basic semialgebraic sets.* Expo. Math. **9** (1991), 289–334.

[Brö2] ________: *Spaces of orderings and semialgebraic sets*. Can. Math. Soc. Conf. Proc. **4** (1984), 231–248.

[Ma1] M. Marshall: *Separating families for semialgebraic sets*. Manuscripta math. **80** (1993), 73–79.

[Ma2] ________: *Quotients and inverse limits of spaces of orderings*. Can. J. Math. **31** (1979), 604–616.

[Rz] J.M. Ruiz: *A note on a separation problem*. Arch. Math. **43** (1984), 422–426.